Comments on non-relativistic AdS/CFT

Yuji Tachikawa (IAS)

in collaboration with

Juan Maldacena & Dario Martelli

[arXiv:0807.1100]

also see [Herzog-Rangamani-Ross,0807.1099] and [Adams-Balasubramanian-McGreevy,0807.1111]

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Nonrelativistic conformal group

$d + 1$ dimensional Galilean group

- M_{ij}: rotation
- P_i: translation
- K_i: Galilean boost, $[P_i, K_j] = -i\delta_{ij} M$, where M: mass

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Conformal extension

- D: dilatation with dynamical exponent z

$$[D, P_i] = -iP_i, \qquad [D, H] = -izH,$$
$$[D, K_i] = i(z-1)K_i \qquad [D, M] = i(z-2)M$$

- C: special conformal transformation when $z=2$

$$[C, P_i] = iK_i, \qquad [D, C] = 2iC, \qquad [H, C] = iD.$$

Nonrelativistic conformal group

Called as Schrödinger group when $z = 2$ because

$$S = \int d^d x dt \left[\psi^\dagger i\partial_t \psi - \frac{1}{2m}(\partial_i \psi)^2 \right]$$

has this symmetry.

- D: $x \to \lambda x$, $t \to \lambda^2 t$

$$D = \int d^d x \, x_i j_i(x), \qquad j_i(x) = i\psi^\dagger \partial_i \psi - i(\partial_i \psi^\dagger)\psi$$

- C: $x \to x/(1-\lambda t)$, $t \to t/(1-\lambda t)$

$$C = \int d^d x \, \frac{x^2}{2} n(x), \qquad n(x) = \psi^\dagger \psi$$

Nonrelativistic conformal system

- Fermion at "unitarity" in $d = 3$ [Mehen-Stewart-Wise]

$$S = \int d^3x dt \left[\psi_\sigma^\dagger i\partial_t \psi_\sigma - \frac{1}{2m}(\partial_i \psi_\sigma)^2 + g(\psi_\downarrow^\dagger \psi_\uparrow^\dagger \ \psi_\downarrow \psi_\uparrow \ ) \right]$$

when "$g \to \infty$" or more precisely at infinite scattering length

- Experimentally realized in the system of trapped cold atoms
- Strongly coupled, hard to solve ⟶ Might AdS/CFT help ???

- Interacting anyon gas in $d = 2$ [Jackiw-Pi,Bergman-Lozano]

Aside: State-Operator correspondence in NR CFT

- doesn't make sense to put the theory on S^3

Aside: State-Operator correspondence in NR CFT

- doesn't make sense to put the theory on S^3
- instead consider changing

$$H \to \tilde{D} \equiv H + C = \int d^d x \left[\epsilon(x) + \frac{x^2}{2} m(x) \right]$$

Cold atoms in the harmonic potential.

- one can show

$$\tilde{D}|\mathcal{O}\rangle = \Delta|\mathcal{O}\rangle$$

where $|\mathcal{O}\rangle \equiv e^{-H}\mathcal{O}|0\rangle$ and $[D, \mathcal{O}] = -\Delta\mathcal{O}$ [Nishida-Son]

Aside: Unitarity bound in NR CFT

Unitarity bound in relativistic CFT

- algebra P_μ, K_μ, $M_{\mu\nu}$, D $\longrightarrow$ $\tilde{P}_i$, $\tilde{K}_i$, $\tilde{M}_{ij}$, $\tilde{D}$.
- Norm of $\tilde{P}_i\tilde{P}_i|\mathcal{O}\rangle = 2\Delta + 2 - d$ if spin 0.
- $\Delta \geq (d-2)/2$.
- saturated = free relativistic particle

Unitarity bound in NR CFT

- algebra P_i, K_i, M_{ij}, H, D, C $\longrightarrow$ $\tilde{P}_i$, $\tilde{K}_i$, $\tilde{M}_{ij}$, $\tilde{H}$, $\tilde{D}$, $\tilde{C}$
- Norm of $(\tilde{H} + \tilde{P}_i\tilde{P}_i/2M)|\mathcal{O}\rangle = 2\Delta - d$.
- irrespective of spin.
- $\Delta \geq d/2$.
- saturated = free Schrödinger.

Contents

Background with non-relativistic conformal symmetry

- [Son] and [Balasubramanian-McGreevy] found the metric

$$ds^2 = -\sigma^2 r^{2z}(dx^+)^2 + \frac{dr^2}{r^2} + r^2(-dx^+dx^- + d\vec{x}^2)$$

  which has the non-relativistic conformal symmetry with dynamical exponent z.
- D acts as follows :

$$\vec{x} \to \lambda\vec{x}, \quad x^+ \to \lambda^z x^+, \quad x^- \to \lambda^{2-z}x^-, \quad r \to r/\lambda.$$

- $x^+ \leftrightarrow H, \qquad x^- \leftrightarrow M$
- Deformation of AdS_{d+2} for the non-relativistic conformal system with d spatial dimension
- Seems natural to compactify x^- when $z = 2$

Discrete Light-Cone Quantization

- If one compactifies x^-, deformation is not necessary ...
- The isometry of

$$ds^2 = +\frac{dr^2}{r^2} + r^2(-dx^+dx^- + d\vec{x}^2), \qquad x^- \sim x^- + r^-$$

is exactly the Schrödinger group.

- DLCQ of relativistic theory $\longrightarrow$ looks like Galilean.

$$p_+p_- - \vec{p}^{\,2} = 0 \longrightarrow E = \frac{\vec{p}^{\,2}}{M} \text{ where } E = p_+,\ M = p_-$$

- DLCQ of relativistic conformal theory $\longrightarrow$ Galilean + conformal !

Discrete Light-Cone Quantization

- $\mathrm{AdS}_5 \times S^5$ with $x^- \sim x^- + r^-$ $\longrightarrow$ DLCQ of $\mathcal{N} = 4$ SYM.
- Mysterious theory in 2+1 dimensions. Doesn't at all look like fermions at unitarity ...
- $\mathrm{AdS}_7 \times S^4$ with $x^- \sim x^- + r^-$ $\longrightarrow$ DLCQ of M5-brane theory

Discrete Light-Cone Quantization

- $\mathrm{AdS}_5 \times S^5$ with $x^- \sim x^- + r^-$ ⟶ DLCQ of $\mathcal{N} = 4$ SYM.
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- $\mathrm{AdS}_7 \times S^4$ with $x^- \sim x^- + r^-$ ⟶ DLCQ of M5-brane theory
- Studied already in [Aharony-Berkooz-Seiberg,'97].
  NR superconformal group was written down there.
- Theory in 4+1 dimensions. N momenta along x^-, k M5-brane
  ⟶ Quantum mechanics of N instantons of $U(k)$ gauge group
- It has 4-d Schrödinger symmetry, but definitely not cold atoms in 4d.

Noncommutative deformation

Deformed version with $z = 2$

$$ds^2 = -\sigma^2 r^4 (dx^+)^2 + \frac{dr^2}{r^2} + r^2(-dx^+ dx^- + d\vec{x}^2)$$

times S^5 can be obtained by a solution-generating technique.
TsT transformation or Melvin twist.

1. Choose a direction φ in S^5. T-dualize φ to $\tilde{\varphi}$
2. redefine new x^- to be $x^-_{\text{new}} = x^-_{\text{old}} + \sigma\tilde{\varphi}$
3. T-dualize $\tilde{\varphi}$ back to φ.

Field theory side: funny non-commutativity

$$f * g = e^{i(P_f R_g - P_g R_f)} fg$$

where P: momentum along x^-, R: R-charge

Noncommutative deformation

So, the background of [Son],[Balasubramanian-McGreevy]
is dual to DLCQ of funny noncommutative deformation of $\mathcal{N} = 4$ SYM.

It looks quite different from cold atoms (=fermion at unitarity.)
One can study its thermodynamic properties e.g. viscosity, entropy with this caveat in mind:

Is the difference larger than that of QCD at RHIC and hot $\mathcal{N} = 4$ SYM ?

Strong coupling necessary for having gravity dual, but not sufficient.

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More caveats

- Vacuum solutions have x^- compactified $\longrightarrow$ sub-stringy length. Can't trust supergravity approx.

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- Vacuum solutions have x^- compactified $\longrightarrow$ sub-stringy length. Can't trust supergravity approx.
- With momenta along x^- and finite temperature,
  $\longrightarrow$ x^- direction becomes spacelike
- Asymptotes to vacuum solutions $\longrightarrow$ still bad at $r \to \infty$
- Thermodynamic properties determined by the horizon $r \sim r_H$
- should be OK. cf. holography for Dp-brane with $p \neq 3$ [Itzhaki-Maldacena-Sonnenschein-Yankielowicz]

BH solution

- AdS → TsT → Schrödinger bkg.
- non-extremal brane solution
  → TsT → finite temperature solution.
- Near-horizon form of the non-extremal D3-brane

$$ds^2 = \frac{1}{1 - r_0^4/r^4}\frac{dr^2}{r^2} + r^2\left[-dx^+dx^- + \frac{r_0^4}{4r^4}\left(\lambda^{-1}dx^+ + \lambda dx^-\right)^2 + d\vec{x}^{\,2}\right]$$

- Horizon at $r = r_0$,
- x^- direction spacelike.

BH solution

$$ds^2 = e^{\frac{3}{2}\Phi} r^2 \left[\left(-1 + \frac{r_0^4}{2r^4}\right) dx^+ dx^- + \frac{r_0^4}{4r^4}\left(\lambda^2 (dx^-)^2 + \lambda^{-2}(dx^+)^2\right) - \sigma^2 r^2 \left(1 - \frac{r_0^4}{r^4}\right)(dx^+)^2 \right] + e^{-\frac{\Phi}{2}} r^2 \left[\frac{1}{r^4 - r_0^4} dr^2 + d\vec{x}^2 \right] + e^{-\frac{\Phi}{2}} ds^2(B_{KE}) + e^{\frac{3}{2}\Phi}\eta^2$$

with dilaton and B-field given by

$$e^{-2\Phi} = 1 + \sigma^2 \lambda^2 \frac{r_0^4}{r^2},$$

$$B = \sigma \frac{r^2}{2} e^{2\Phi} \left[\left(2 - \frac{r_0^4}{r^4}\right) dx^+ - r_0^4 \lambda^2 \, dx^- \right] \wedge \eta.$$

Temperature, etc.

- Energy: $E = -P_+$, Particle Number: $N = r^-(-P_-)$;
- Temperature: surface gravity; Entropy: area law
- chemical potential μ : g_{+-} component at the horizon

$$N/V \propto (r^-)^2\lambda^2 r_0^4, \qquad E/V \propto r^- r_0^4$$
$$T \propto r_0/\lambda \qquad \mu = 1/(r^-\lambda^2)$$
$$S/V \propto \lambda r^- r_0^3.$$

- Satisfy the 1st law, $\delta E = T\delta S - \mu\delta N$.
- $E \propto T^4/\mu^2$, $\qquad E/N \propto \mu$

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- Why are they so simple ?

Why so simple ?

- $E \propto T^4/\mu^2, \qquad E/N \propto \mu$
- Dilatation $x \to kx$, $t \to k^2 t$ should transform

$$T \to k^{-2}t, \qquad E/V \to k^{-4}(E/V), \qquad \mu \to k^{-2}\mu$$

- $E/V = T^2 f(\mu/T)$.
- Why $f(x) = x^{-2}$ for supergravity solutions ??
  (n.b. different from free theories)

Why so simple ?

- Another 'solution generating technique':

  just boost $x^+ \to \lambda x^+$, $x^- \to x^-/\lambda$, but keep r^- fixed.

$$T \to T/\lambda, \qquad E/V \to E/V, \qquad \mu \to \mu/\lambda^2$$

- $E/V = g(\mu/T^2)$.

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- Another 'solution generating technique':

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$$T \to T/\lambda, \qquad E/V \to E/V, \qquad \mu \to \mu/\lambda^2$$

- $E/V = g(\mu/T^2)$.
- $E/V = T^2 f(\mu/T) \longrightarrow E/V = T^4/\mu^2$
- Universal prediction of having a weakly-curved gravity dual.

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Deformed background

The metric

$$ds^2 = -\sigma^2 r^{2z}(dx^+)^2 + \frac{dr^2}{r^2} + r^2(-dx^+dx^- + d\vec{x}^2)$$

is not Einstein, but a solution of the system

$$S = \int d^{d+3}x\sqrt{-g}\left[R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu\right]$$

with $A \propto r^z dx^+$.

$\Lambda = -(d+1)(d+2)/2, \qquad m^2 = z(z+d).$

Is it possible to embed it to 10d/11d supergravity ?

Massive fields

Needs massive fields $m^2 = z(z + d)$ in the reduction.

In $\mathrm{AdS}_5 \times \mathrm{SE}^5$ compactification,
one has the Reeb 1-form η and 2-form $\omega = d\eta$.

For S^5, think of it as S^1 bundle parametrized by φ over $\mathbb{CP}^2$.
Then $\eta = d\varphi + \cdots$ and ω: Kähler form of $\mathbb{CP}^2$.

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$B_2 = A \wedge \eta$ ⟶ A has $m^2 = 8$ ⟶ $z = 2$.

$C_4 = \mathbb{A} \wedge \star\omega$ and $ds^2(\mathrm{SE}) = (d\varphi + \mathcal{A})^2 + \cdots$
They mix ⟶ Modes with $m_+^2 = 24$ and $m_-^2 = 0$. ⟶ $z = 4$.

Non-linear reduction with $m^2 = 8$

Ansatz:

$$ds_{10}^2 = e^{-\frac{2}{3}(4U+V)} ds^2(M) + e^{2U} ds^2(B_{\mathrm{KE}}) + e^{2V}\eta^2 ,$$
$$B = A \wedge \eta + \theta\,\omega ,$$
$$F_5 = (1+\star)G_5 \qquad \text{where} \quad G_5 = 4e^{-4U-V}\,\mathrm{vol}(M)$$

where $ds^2(B_{KE}) + \eta^2$ is a Sasaki-Einstein metric.
Nontrivial dilaton; other fields zero.

$$S = \frac{1}{2}\int d^5x\sqrt{-g}\Big[R + 24e^{-u-4v} - 4e^{-6u-4v} - 8e^{-10v} - 5\partial_a u\partial^a u - \frac{15}{2}\partial_a v\partial^a v - \frac{1}{2}\partial_a\Phi\partial^a\Phi - \frac{1}{4}e^{-\Phi+4u+v}F_{ab}F^{ab} - 4e^{-\Phi-2u-3v}A_aA^a\Big] ,$$

where $u = (2/5)(U-V)$ and $v = (4/15)(4U+V)$
This is a consistent reduction!

Non-linear reduction with $m^2 = 8$

Ansatz:

$$ds^2_{10} = e^{-\frac{2}{3}(4U+V)} ds^2(M) + e^{2U} ds^2(B_{\mathrm{KE}}) + e^{2V} \eta^2 ,$$

$$B = A \wedge \eta + \theta\, \omega ,$$

$$F_5 = (1+\star) G_5 \qquad \text{where} \quad G_5 = 4 e^{-4U-V}\, \mathrm{vol}(M)$$

- Can't turn on B alone.
- need to keep U: size of the base, V: size of the fiber, Φ: dilaton
- θ eaten by A to become massive
- Action with θ, U, V, and Φ given in [Klebanov-Tseytlin] for S^5, [Benvenuti-Mahato-YT-Pando Zayas] for SE_5

Non-linear reduction with $m^2 = 24$

Ansatz:

$$ds^2 = e^{-\frac{2}{3}(4U+V)} ds^2(M) + e^{2U} ds^2(B_{\mathsf{KE}}) + e^{2V}(\eta + \mathcal{A})^2 ,$$

$$F_5 = (1 + \star_{10})\big[2\omega^2 \wedge (\eta + \mathcal{A}) + 2\omega^2 \wedge (\mathbb{A} - \mathcal{A}) - \omega \wedge (\eta + \mathcal{A}) \wedge \mathbb{F}\big]$$

where $\mathbb{F} = d\mathbb{A}$, $\mathcal{F} = d\mathcal{A}$.

$$S = \frac{1}{2}\int d^5x\sqrt{-g}\Big[R + 24e^{-u-4v} - 4e^{-6u-4v} - 8e^{-10v} - 5\partial_a u \partial^a u - \frac{15}{2}\partial_a v \partial^a v - \frac{1}{4}e^{-4u+4v}\mathcal{F}_{ab}\mathcal{F}^{ab} - \frac{1}{2}e^{2u-2v}\mathbb{F}_{ab}\mathbb{F}^{ab} - 8e^{-4u-6v}(\mathbb{A}-\mathcal{A})_a(\mathbb{A}-\mathcal{A})^a\Big] + \frac{1}{2}\int \mathcal{A}\wedge\mathbb{F}\wedge\mathbb{F} ,$$

Again, this is a consistent reduction!

Contents

Summary

Done

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- Thermodynamics
- Consistent truncation with massive fields.

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To do

- Extract more physics.
- What are the dual field theories ?
- How different are they from cold atoms ?